Will a random walker return home?

Solution using electric networks

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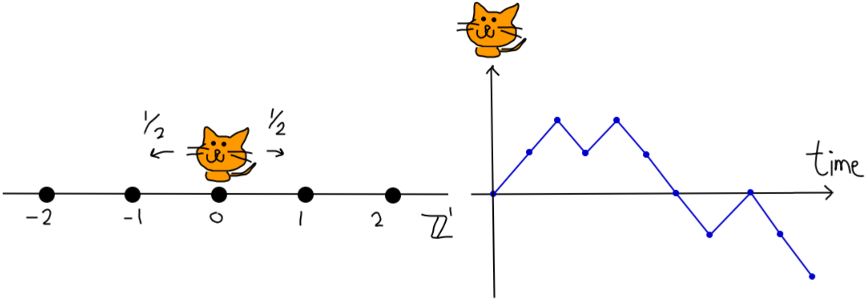
Simple random walk on $\mathbb{Z}^1$

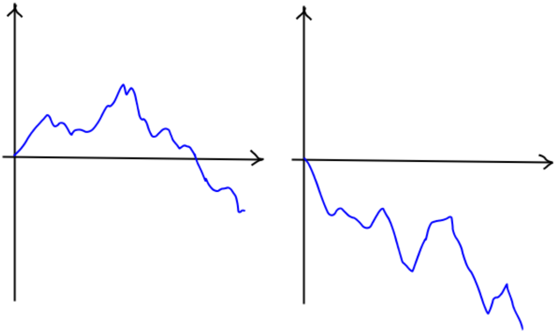
The cat starts at 0 and jumps randomly.



What is the probability that the cat never return to the starting position?

Simple random walk on $\mathbb{Z}^1$

Cat hits 0 | Cat avoids 0 (so far)



What is $\mathbf{P}_0\{\text{cat avoids 0 forever}\}$?

Theorem 1. $\mathbf{P}_0\{\text{cat avoids 0 forever}\} = 0.$

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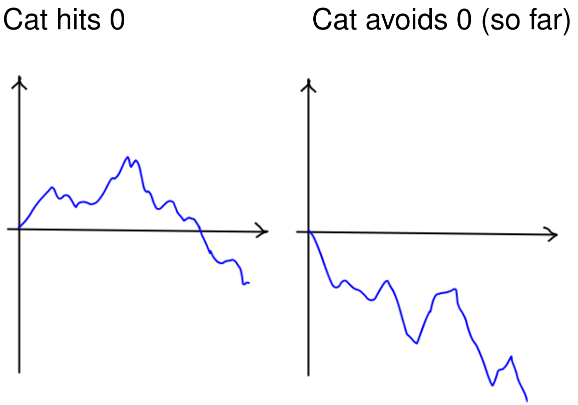

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Theorem 1. $\mathbf{P}_0\{\text{cat avoids 0 forever}\} = 0.$

Theorem 1 is equivalent to the gambler's ruin

After one step, the cat will be at position 1 or position -1 (each has probability $\frac{1}{2}$).

Conditioning on the first step,

$$\begin{aligned} & \mathbf{P}_0\{\text{cat never returns to } 0\} \\ = \; & \frac{1}{2}\mathbf{P}_1\{\text{cat never visits } 0\} + \frac{1}{2}\mathbf{P}_{-1}\{\text{cat never visits } 0\}. \end{aligned}$$

By symmetry,

$$\mathbf{P}_1\{\text{cat never visits } 0\} = \mathbf{P}_{-1}\{\text{cat never visits } 0\}.$$

Hence

$$\mathbf{P}_0\{\text{cat never returns to } 0\} = \mathbf{P}_1\{\text{cat never visits } 0\}.$$

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Gambler's ruin

Start with \$1. Gamble with a fair but infinitely rich guy.

Then $\mathbf{P}_1\{\text{cat never visits } 0\} = \mathbf{P}_1\{\text{you never bankrupt}\}$.

Theorem 1 implies you will surely go bankrupt.

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Proof of Theorem 1

Aim: $\mathbf{P}_1\{\text{cat never visits } 0\} = 0$.

For $x \geq 0$, let

$$u(x) = \mathbf{P}_x\{\text{cat never visits } 0\}. \text{ (starting at } x\text{)}$$

Then $u(0) = 0$ (since it starts at 0), and conditioning on the first step,

$$u(x) = \frac{1}{2}u(x-1) + \frac{1}{2}u(x+1), \quad x \geq 1.$$

(u is *discrete harmonic* on $\mathbb{Z}^+$.)

$$\Rightarrow u(x) - u(x-1) = u(x+1) - u(x).$$

So u has constant increment. But $0 \leq u(x) \leq 1$. So $u \equiv 0$. $\square$

We have shown in fact that $\mathbf{P}_x\{\text{cat never visits } 0\} = 0$ for all x.

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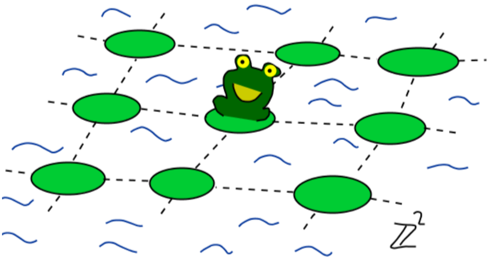
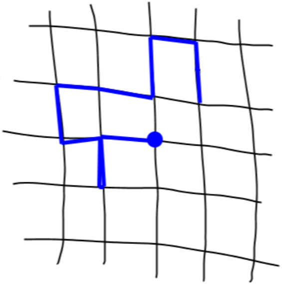
Simple random walk on $\mathbb{Z}^2$.

The frog starts at $(0, 0)$.

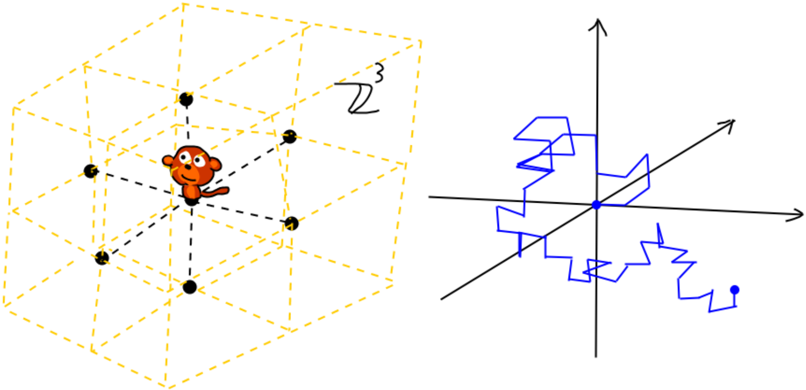




Will the frog return to the starting leaf?

Simple random walk on $\mathbb{Z}^3$.

The monkey starts at $(0, 0, 0)$.



Will the monkey return to the starting point?

Main theorem

Theorem 2 (Polya 1921). Consider the simple random walk on $\mathbb{Z}^d$, starting at the origin.

- If $d = 1$ or 2, the random walker will return to the starting point with probability 1. *(recurrent)*
- For $d \geq 3$, there is a positive probability that the random walker will not return to the starting point. *(transient)*

The *electric network approach* is due to Nash-Williams (1959). Our presentation follows Doyle (1994).

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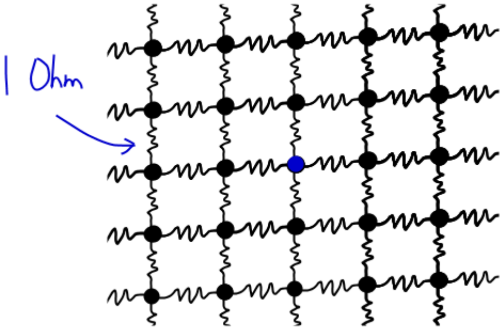
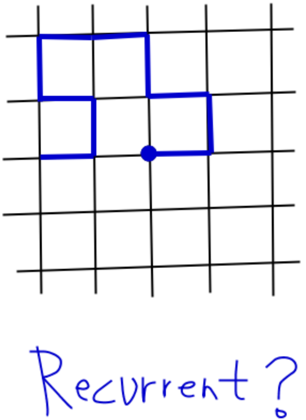
Connection with electric networks

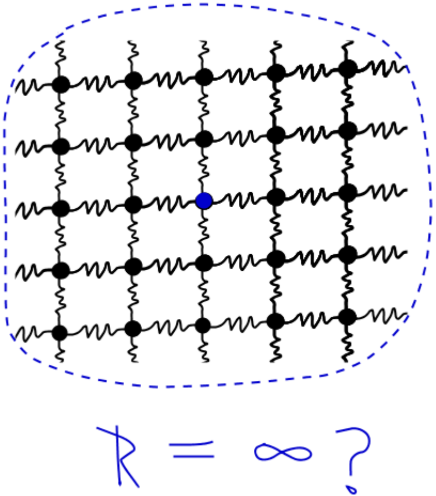
Construct a network according to the lattice (the figure is the network for $\mathbb{Z}^2$). Put a unit resistor on each edge.



Connection with electric networks

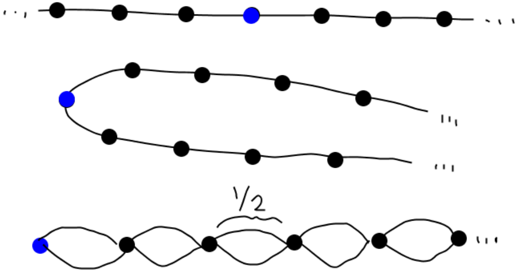
Claim: walk is recurrent $\Leftrightarrow$ effective resistance is ∞





Connection with electric networks

Theorem 1 becomes an easy corollary.



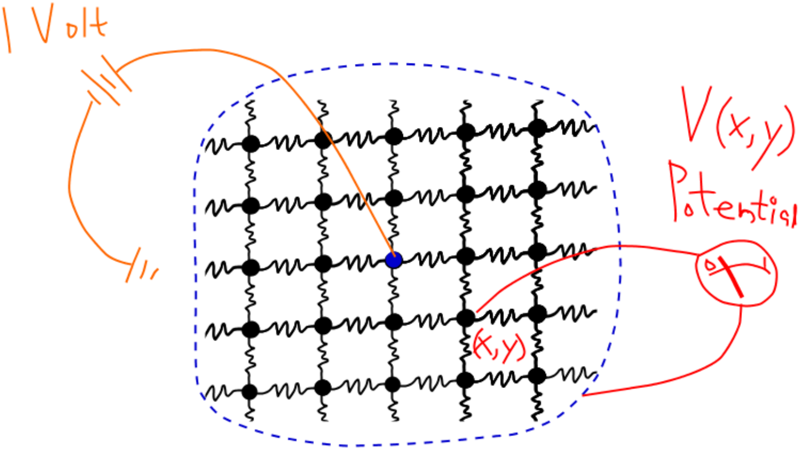
Resistance from 0 to 'infinity' is

$$R = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

Hence the walk on $\mathbb{Z}^1$ is recurrent.

Proof of claim (network)

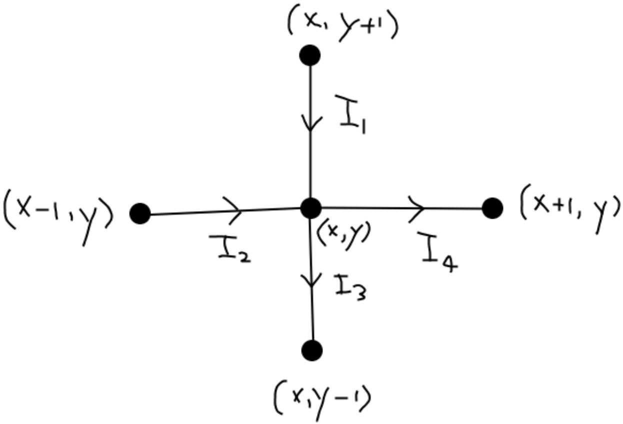
Consider a large square containing $(0, 0)$. Ground the boundary. Put a unit battery.



$V(x, y)$ is the potential at the node (x, y). $V(0, 0) = 1$.
$V(x, y) = 0$ on the boundary.

Proof of claim (network)

Consider an interior node $\neq (0,0)$. *Ohm's law*: $V = IR$



$I_1 = V(x, y+1) - V(x, y)$, $I_2 = V(x-1, y) - V(x, y)$,
$I_3 = V(x, y) - V(x, y-1)$, $I_4 = V(x, y) - V(x+1, y)$.

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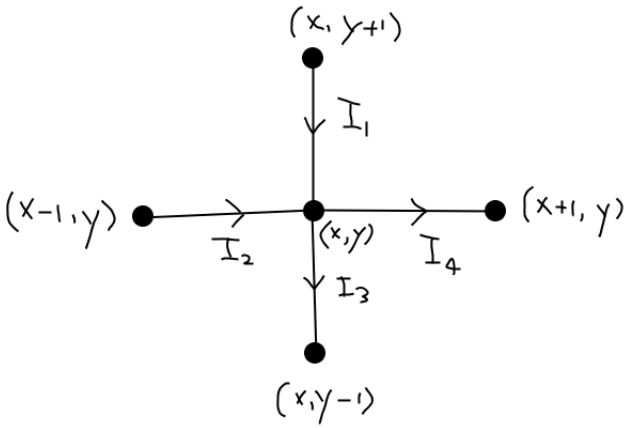

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Proof of claim (network)

current in = current out: $I_1 + I_2 = I_3 + I_4$. Hence

$$\begin{aligned} & V(x,y+1) - V(x,y) + V(x-1,y) - V(x,y) \\ = \; & V(x,y) - V(x,y-1) + V(x,y) - V(x+1,y) \end{aligned}$$

So V has the *averaging property*:

$$V(x,y) = \frac{1}{4}(V(x-1,y) + V(x,y+1) + V(x+1,y) + V(x,y-1)).$$

(Call V *discrete harmonic*.) Thus

$$\begin{cases} V(0,0) = 1 \\ V(x,y) = 0 \text{ on boundary} \\ V \text{ is discrete harmonic on interior} \setminus \{(0,0)\} \end{cases}$$

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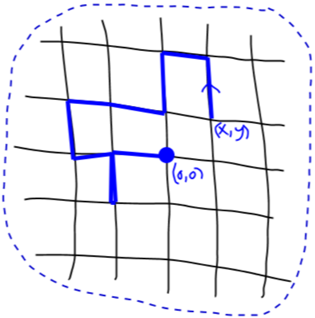
Proof of claim (probability)

Let

$$U(x,y) = \mathbf{P}_{(x,y)}\{\text{frog visits } (0,0) \text{ before hitting boundary}\}.$$



Then $U(0,0) = 1$ and $U(x,y) = 0$ on boundary.

Proof of claim (probability)

On interior $\setminus\{(0,0)\}$, conditioning on the first step,

$$U(x,y) = \frac{1}{4}(U(x-1,y)+U(x,y+1)+U(x+1,y)+U(x,y-1)).$$

We conclude:

$$\begin{cases} U(0,0) = 1 \\ U(x,y) = 0 \text{ on boundary} \\ U \text{ is discrete harmonic on interior} \setminus \{(0,0)\} \end{cases}$$

Hence U and V satisfies the same equations!

$$\Rightarrow U(x,y) = V(x,y) \textit{ (uniqueness of solution)}$$

Probability is the voltage!

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$$\mathbb{P}_{(0,0)}\{\text{frog never returns to } (0,0) \text{ before hitting boundary}\}$$

$$= 1 - \mathbb{P}_{(0,0)}\{\text{frog returns to } (0,0) \text{ before hitting boundary}\}$$

$$= 1 - \frac{1}{4}(U(-1,0) + U(1,0) + U(0,1) + U(0,-1))$$

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$$= \frac{1}{4}[(1-V(-1,0)) + (1-V(1,0)) + (1-V(0,1)) + (1-V(0,-1))]$$

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$$= 1 - \frac{1}{4}(V(-1,0) + V(1,0) + V(0,1) + V(0,-1))$$

$$= \frac{1}{4}[(1-V(-1,0))+(1-V(1,0))+(1-V(0,1))+(1-V(0,-1))]$$

$$= \frac{1}{4}\text{current coming out from } (0,0)$$

$$= \frac{1}{4 \times \text{effective resistance between } (0,0) \text{ and boundary}}.$$

Proof of claim (together)

$$\mathbb{P}_{(0,0)}\{\text{frog never returns to } (0,0) \text{ before hitting boundary}\}$$

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$$= 1 - \frac{1}{4}(U(-1,0) + U(1,0) + U(0,1) + U(0,-1))$$

$$= 1 - \frac{1}{4}(V(-1,0) + V(1,0) + V(0,1) + V(0,-1))$$

$$= \frac{1}{4}[(1 - V(-1,0)) + (1 - V(1,0)) + (1 - V(0,1)) + (1 - V(0,-1))]$$

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Proof of claim (together)

Letting the boundary farther and farther away,

$$\begin{aligned} & \mathbf{P}_{(0,0)}\{\text{frog never returns to } (0,0)\} \\ = \; & \frac{1}{4 \times \text{effective resistance between } (0,0) \text{ and `infinity'}} \\ = \; & \frac{1}{4R}. \quad \square \end{aligned}$$

Note: On $\mathbb{Z}^1$, we have

$$\mathbf{P}_0\{\text{cat never returns to } 0\} = \frac{1}{2R}.$$

On $\mathbb{Z}^3$, we have

$$\mathbf{P}_{(0,0,0)}\{\text{monkey never returns to } (0,0,0)\} = \frac{1}{6R}.$$

Proof of claim (together)

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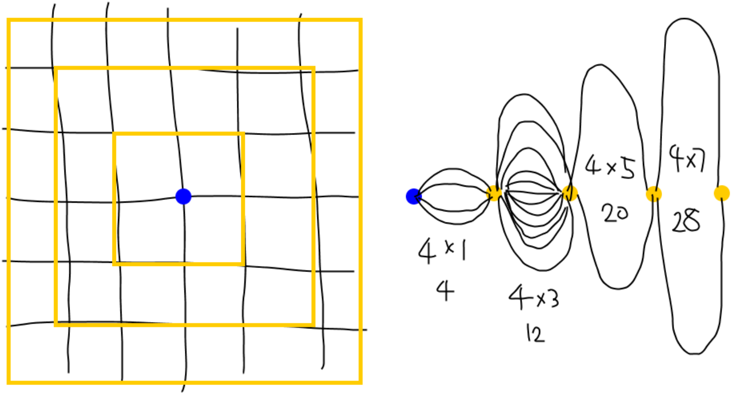
$$\mathbf{P}_{(0,0,0)}\{\text{monkey never returns to } (0,0,0)\} = \frac{1}{6R}.$$

Effective resistance for $\mathbb{Z}^2$ is infinite!

Replace the yellow edges with wires (*shorting*). Effective resistance decreases.



$$R \geq \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{2n+1} = \infty.$$

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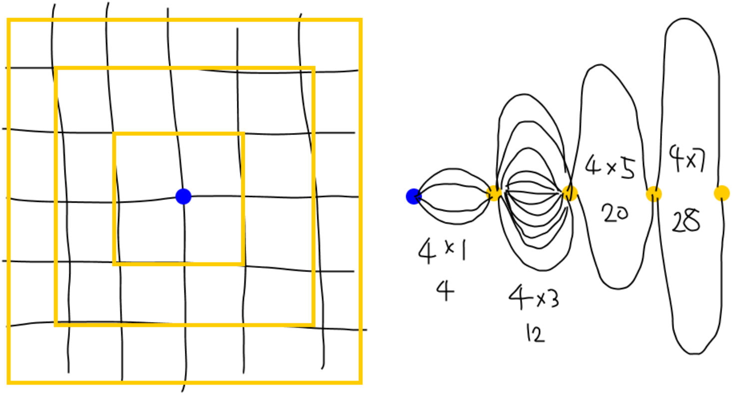
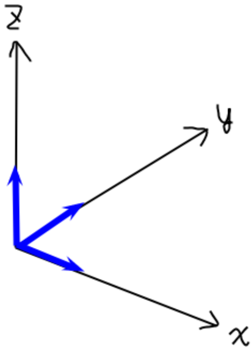


$$R \geq \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{2n+1} = \infty.$$

Effective resistance for $\mathbb{Z}^3$ is finite!

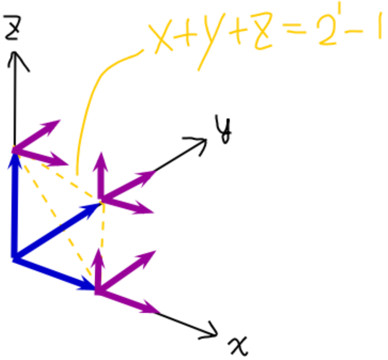
Idea: If we cut away edges, resistance increases. Thus, if we can find a subnetwork with finite resistance, the whole $\mathbb{Z}^3$ has finite resistance.

Construction: Start with 3 rays starting at the origin.



Effective resistance for $\mathbb{Z}^3$ is finite!

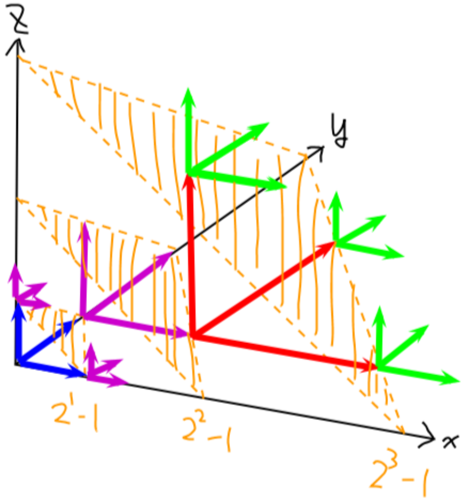
When a ray hits the plane $x + y + z = 2^1 - 1$, it splits into 3 rays:



Continue: When each ray hits the plane $x + y + z = 2^n - 1$, it splits into 3 rays.

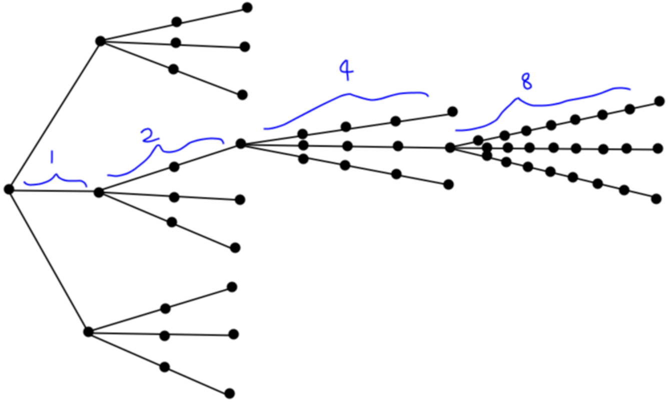
Effective resistance for $\mathbb{Z}^3$ is finite!

The desired subnetwork is the traces of all the rays.



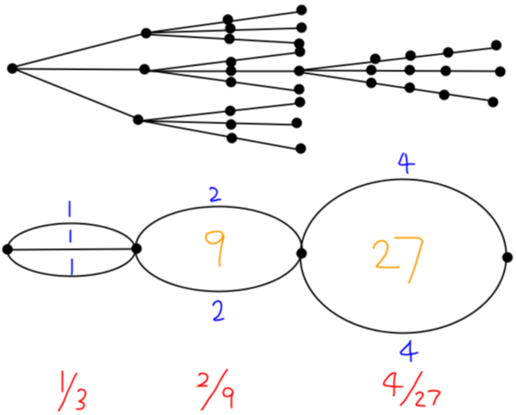
Effective resistance for $\mathbb{Z}^3$ is finite!

The subnetwork can be compared with a tree:



Effective resistance for $\mathbb{Z}^3$ is finite!

Resistance of the tree:



$$\text{Resistance} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) < \infty$$

Further directions

- Discrete harmonic functions ($U(x, y) =$ average) $\leftrightarrow$ harmonic functions ($\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$)
- Gambler's ruin and martingales, financial mathematics
- Simple random walk and Brownian motion
- Random walk on trees and graphs

References and further readings

W. Feller, *An introduction to probability theory and its applications, Volume 1*.

P. G. Doyle and J. L. Snell, *Random walks and electric networks*.

P. G. Doyle, *Application of Rayleigh's short-cut method to Polya's recurrence problem*.

D. A. Levin, Y. Peres, E. L. Wilmer, *Markov chains and mixing times*.