Fun with spheres

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Why spheres?

1. The simplest yet the most symmetrical among closed and bounded surfaces (compare with doughnut)
2. Abundance in nature (earth, soap bubbles...)

 

3. My favorite

What is a sphere?

Definition

A sphere is the set of points in $\mathbb{R}^3$ which all are at a fixed distance R (radius) from a given point (center). Its equation is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 = R^2$$

More generally, the equation for n-dimensional sphere:

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_{n+1} - a_{n+1})^2 = R^2, (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$$

The above is the definition of 2-dimensional sphere, whereas a 1-dimensional sphere is simply a circle.
In this talk 'sphere' always means 2-dimensional sphere, denoted by S^2.
An n-dimensional sphere is denoted by S^n.


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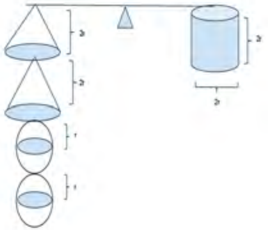
Geometry of spheres

Archimedes of Syracuse found that

$$V = \frac{4}{3}\pi R^3$$

$$A = 4\pi R^2$$

He found out the formula of volume of spheres by an ingenious use of his mechanical theory of lever and Euclidean geometry.



Geometry of spheres

He also found that the surface area is equal to the area of the curved surface of the cylinder which inscribes the sphere.

More is true: The area of any region on a sphere equals that of the region on the cylinder inscribing the sphere, obtained by horizontally projecting the former region onto the cylinder.

This area preserving property of the horizontal projection has found applications in map-making. A world map drawn using this horizontal projection method has the areas of different countries in proportion.

Geometry of spheres

Theorem (Isoperimetric Theorem)

Among all closed and bounded surfaces enclosing a fixed volume, the sphere has the least surface area.

This result has its manifestation in nature.

Geometry of spheres

Let $V_n(R)$ be the volume enclosed by an $(n-1)$-dimensional sphere of radius R. Using some freshman calculus,

$$V_n(R) = \frac{2\pi R^2}{n} V_{n-2}(R)$$

So

$$V_{2k}(R) = \frac{\pi^k R^{2k}}{k!}$$
$$V_{2k+1}(R) = \frac{2^{k+1}\pi^k R^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$$

Question: Can you give a heuristic explanation why the volume of an n-dimensional sphere decreases to zero as $n \to \infty$?

Elements of topology

In geometry, one cares about the notion of distances (metric) and hence angle, sizes, etc.

Let us now turn to another aspect of spheres without regard to distances, sizes, angle, etc.

One may ignore those notions and just care about only the properties of spaces preserved under continuous, bijective transformations (stretching, bending and pinching, without pasting, cutting or puncturing). The relevant field of study in math is *topology*. It is dubbed 'the rubber-sheet geometry'.

Elements of topology

To topologists,

1. 'B' and '8' are the same but different from 0.
2. A doughnut and a mug are indistinguishable, but they are different from a basketball.

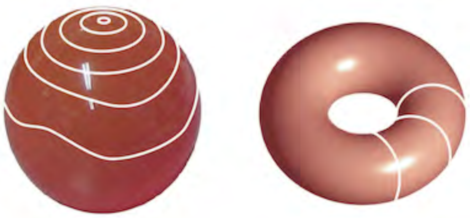
3. A fully inflated basketball is indistinguishable from a deflated one.
4. A fully inflated basketball is different from one with a hole cut open.

Elements of topology

Intuitively, there are no 'holes' on a sphere, whereas there is a 'hole' in a doughnut.

Yet another way to tell the difference between a sphere and a doughnut (a torus):

Any rubber band on a sphere can be shrunk to a point continuously, but some rubber band on a doughnut cannot be without tearing or leaving the surface.



Topology of spheres

Theorem (Borsuk-Ulam Theorem)

Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$.

What it means in meteorology: At any moment there is always a pair of antipodal points on the earth with the same temperature and atmospheric pressure!

Topology of spheres

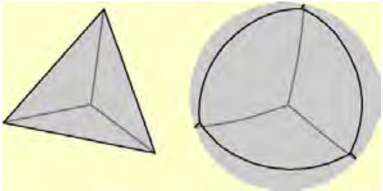
Definition

1. The *Euler characteristic* of a surface S is
$$\chi(S) = V - E + F$$
where V, E and F are the number of vertices, edges and faces of a triangulation of S.
2. Similarly, for a higher dimensional space X,
$$\chi(X) = F_0 - F_1 + F_2 - F_3 + \cdots$$
where F_i is the number of i-th dimensional faces of a triangulation of X.

Topology of spheres

Using a tetrahedron as the triangulation of the sphere S^2,

$$\chi(S^2) = 4 - 6 + 4 = 2.$$



We also have

$$\chi(torus) = 0$$
$$\chi(S^{2n}) = 2$$
$$\chi(S^{2n+1}) = 0$$

Topology of spheres

Question: What are the possible integers $m > 1$ such that there exists a continuous map $f : S^2 \to S^2$ with

1. $\underbrace{f(f(\cdots f(x)))}_{m} = x$ for all $x \in S^2$, and
2. $x, f(x), f(f(x)), \cdots, \underbrace{f(f(\cdots f(x)))}_{m-1}$ are pairwise distinct?

If this map f exists, we may identify, for each $x \in S^2$, the points $x, f(x), f(f(x)), \cdots, \underbrace{f(f(\cdots f(x)))}_{m-1}$ to get another surface P.

Then there is an m-to-1 continuous map from S^2 to P, and

$$\chi(S^2) = m\chi(P)$$

So m can only be 2, f can be taken to be the antipodal map $x \mapsto -x$, and P in this case is the real projective plane.

Topology of spheres

This result can be generalized to any even dimensional sphere, because

$$\chi(S^{2n}) = 2$$

How about the torus? Odd dimensional sphere? It turns out for any $m > 1$, such a map f exists.

For torus, f can be rotation along the longitude (or latitude) by $\dfrac{2\pi}{m}$.

For S^{2n+1}, any points $(x_1, \cdots, x_{2n+2})$ on it can be regarded as $(z_1, \cdots, z_{n+1}) \in \mathbb{C}^n$, where $z_j = x_{2j-1} + ix_{2j}$. One can take

$$f(z_1, z_2, \cdots, z_{n+1}) = (e^{\frac{2\pi i}{m}} z_1, \cdots, e^{\frac{2\pi i}{m}} z_{n+1})$$

Topology of spheres

Definition

Consider a family of maps $g_\theta : S^2 \to S^2$ parametrized by the angle θ of the circle satisfying

1. $g_0(x) = x$ for all $x \in S^2$,
2. $g_{\theta+2\pi}(x) = g_\theta(x)$ for all $x \in S^2$,
3. $g_{\theta_1+\theta_2}(x) = g_{\theta_1}(g_{\theta_2}(x))$ for all $x \in S^2$, and
4. fixing any point $x_0 \in S^2$, if we vary θ from 0 to 2π, then $g_\theta(x_0)$ traces out either a simple closed smooth curve or just a single point x_0.

We call this family of maps g_θ *a smooth circle action* on S^2.

Question: Is there a point $x_0 \in S^2$ such that $g_\theta(x_0) = x_0$ for all θ, i.e. a fixed point for any given smooth circle action? What about a torus, S^{2n} and S^{2n+1}?

Topology of spheres

If we assign a tangent vector to each point of of a surface S in a smooth manner, then we get a *vector field*.

Example

At any moment, the wind distribution on the earth gives rise to a vector field.

Some very special vector fields can be seen as the 'derivative' of some smooth circle action. More precisely, the tangent vectors in those vector fields are tangents of the curves $g_\theta(x_0)$ (if $g_\theta(x_0)$ is a constant curve x_0, the tangent is 0).

Topology of spheres

Definition

A point is called a *zero point* of a vector field if the tangent vector at that point is 0.

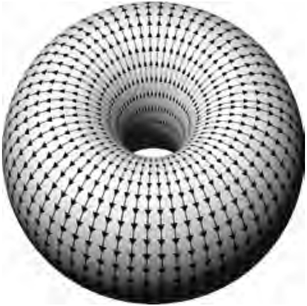
Theorem (Hairy Ball Theorem)

Given any vector field on S^2, there always exists a zero point.

No matter how you comb the hair on a sphere, there is always some hair standing on its end!

Topology of spheres

At any moment, there is always a place on the earth with no wind!



On a torus, one can construct a vector field such that at each point the tangent vector is not zero. What would happen if the earth were a torus?

Question: How about vector fields on S^{2n}? S^{2n+1}?

Topology of spheres

Definition

If x_0 is an isolated zero point of a vector field on a surface S, then the *index* of x_0 is the number of times the tangent vectors goes around you as you go around a simple closed curve enclosing x_0 counterclockwise.

Theorem (Poincaré-Hopf Theorem)

For any vector field with isolated zero points on S,

$$\textit{Sum of indices of all zero points} = \chi(S).$$

If there is a vector field on S without zero points, then $\chi(S) = 0$.

Poincaré-Hopf Theorem implies Hairy Ball Theorem, as $\chi(S^2) = 2 \neq 0$.

Higher dimension analogue of the above theorem shows that $\chi(\text{torus}) = 0$ and $\chi(S^{2n+1}) = 0$.

Topology of spheres

Let g_θ be rotation of a sphere around an axis. There are two fixed points, namely, the north pole and the south pole. The curves $g_\theta(x_0)$ are latitudes and the two poles.

Definition

A circle action h_θ on the tangent vectors of S^2 is a smooth map satisfying

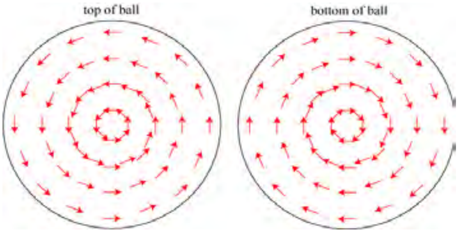
1. $h_0(v_x) = v_x$,
2. $h_\theta(v_x)$ is a tangent vector at $g_\theta(x)$,
3. $h_{\theta_1}(h_{\theta_2}(v_x)) = h_{\theta_1+\theta_2}(v_x)$, and
4. $h_\theta = h_{\theta+2\pi}$.

Topology of spheres

Let v_N and v_S be fixed tangent vectors at the north and south poles respectively. Then $h_\theta(v_N)$ and $h_\theta(v_S)$ are tangent vectors at the two poles.

Definition

Let I_N (resp. I_S) be the number of times $h_\theta(v_N)$ (resp. $h_\theta(v_S)$) goes around the north pole (resp. south pole) as θ varies from 0 to 2π.



Topology of spheres

Theorem

$$I_N - I_S = 2 = \chi(S^2)$$

for any circle action h_θ.

This result and Poincaré-Hopf Theorem are consequences of Atiyah-Singer Index Theorem, a far-reaching result in mathematics.

Atiyah-Singer Index Theorem

1. Tools: vector bundles,'linear algebra' in topology, K-theory, characteristic classes, partial differential equations.
2. Analytic index=Topological index
3. In the above theorems,

$$\begin{aligned} \text{topological index} &= \text{Euler characteristic} \\ \text{analytic index} &= \text{sum of indices of zero points and } I_N - I_S \end{aligned}$$

Atiyah-Singer Index Theorem