3. Möbius Transformations

3.1. Cross Ratio. We have already known that there is geometric meaning to complex numbers, no matter they are considered as points in the plane or special transformations. As a matter of fact, there are many interesting results using complex numbers to study properties related with circles.

We all know from basic geometry that one can always draw a straight line through two given points; while a circle through three given points. But, we need additional conditions to draw straight line through three points and circle through four points. These conditions can be simplified in complex numbers.

Let \( z_1, z_2, z_3 \in \mathbb{C} \) be distinct points; \( L_1 \) and \( L_2 \) be the line segments from \( z_3 \) to \( z_1 \) and to \( z_2 \) respectively.

\[
\begin{align*}
\angle z_1 z_3 z_2 &= \text{arg} \left( \frac{z_1 - z_3}{z_2 - z_3} \right) \\
\text{condition for the three points to be collinear is } \phi &= 0 \text{ or } \phi = \pi.
\end{align*}
\]

Then, the set \( \text{arg} \left( \frac{z_1 - z_3}{z_2 - z_3} \right) \) contains angles (represented by \( \phi \)) from \( L_2 \) to \( L_1 \). The condition for the three points to be collinear is \( \phi = 0 \) or \( \phi = \pi \). On the other hand, it is clear that \( \frac{z_1 - z_3}{z_2 - z_3} \in \mathbb{R} \) if and only if

\[
0 \text{ or } \pm \pi \in \text{arg} \left( \frac{z_1 - z_3}{z_2 - z_3} \right) = \begin{cases} 2k\pi : k \in \mathbb{Z} & = \{ \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots \} \\ (2k + 1)\pi : k \in \mathbb{Z} & = \{ \ldots, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots \} \end{cases}.
\]

Thus, three points \( z_1, z_2, z_3 \) are collinear if and only if \( \frac{z_1 - z_3}{z_2 - z_3} \in \mathbb{R} \). In the following, one will see that it is better to put in a fourth point \( z_4 \), which may be \( \infty \).
To consider points in the extended plane $\mathbb{C}$, we need the following conventions about arithmetics on $\mathbb{C}$. For any complex number $z \in \mathbb{C}$,

$$z \pm \infty = \infty, \quad z \cdot \infty = \infty, \quad \frac{\infty}{z} = \infty, \quad \frac{z}{\infty} = 0,$$

where $z \neq 0$,

$$\lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c} \text{ if } \frac{a}{c} \text{ defined.}$$

With this convention, our calculation above can be rewritten as

$$z_1 z_3 = z_2 z_4 = z_1 z_3 = z_2 z_4.$$

This value belongs to $\mathbb{R}$ if and only if the three points and $\infty$ are on a straight line.

**Definition 3.1.** Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points on the Riemann sphere.

The cross ratio is

$$[z_1, z_2, z_3, z_4] \overset{def}{=} \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

**Exercise 3.1.** Find the cross ratio $[z_1, z_2, z_3, z_4]$ where

<table>
<thead>
<tr>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>-1 - i</td>
<td>0</td>
<td>2 + 2i</td>
<td>5 + 5i</td>
<td>1</td>
<td>3 + i</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>-1 - i</td>
<td>0</td>
<td>5 + 5i</td>
<td>1</td>
<td>3 + i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>-1 - i</td>
<td>0</td>
<td>5 + 5i</td>
<td>1</td>
<td>3 + i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>1 + i</td>
<td>-1 + i</td>
<td>1 - i</td>
<td>1 - i</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>1 + i</td>
<td>-1 + i</td>
<td>1 - i</td>
<td>1 - i</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(f)</td>
<td>1 + i</td>
<td>-1 + i</td>
<td>1 - i</td>
<td>1 - i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(g)</td>
<td>1 + i</td>
<td>-1 + i</td>
<td>1 - i</td>
<td>1 - i</td>
<td></td>
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</table>

By an argument similar to the above, when one of them is $\infty$, the remaining three points are collinear if and only if the cross ratio is real. This naturally arouses the interest on the situation when none of them is $\infty$. We have

$$\arg[z_1, z_2, z_3, z_4] = \arg\left(\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}\right) = \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) - \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right).$$

Therefore, $[z_1, z_2, z_3, z_4] \in \mathbb{R}$ if and only if $0$ or $\pm \pi \in \arg[z_1, z_2, z_3, z_4]$, which is equivalent to

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right) \quad \text{or} \quad \pi \in \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) - \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right).$$

Let us examine the geometric situation for these two equalities. The straight line joining $z_1$ and $z_2$ always separates the plane into two sides. Assume that neither $z_3$ nor $z_4$ is collinear with $z_1$ and $z_2$. It is because the two equalities can never hold if
$z_3$ or $z_4$ falls on the straight line joining $z_1$ and $z_2$. Then these two points $z_3$ and $z_4$ either both lie on the same side or they lie on different sides. If they lie on different sides, by considering the signs of the principle argument, one has

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) \neq \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right).$$

So the first equality only corresponds to the case that they lie on the same side. Similarly, the second one corresponds to the case of different sides. This leads to the following two pictures in which the two angles are equal on the left while the sum of the two angles is $\pi$ on the right. In both cases, all four points are concyclic.

It is not difficult to prove that this picture also implies the two equalities of arguments. We have the following conclusion.

**Theorem 3.2.** Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points. They lie on a straight line or a circle in the plane if and only if their cross ratio is a real number.

As we have mentioned before, a straight line in the plane corresponds to a circle passing through the north pole under the stereographic projection. Therefore, the cross ratio is telling us whether four points in the Riemann sphere are concyclic.

**Exercise 3.2.** (1) Let $z_1, z_2, z_3$ be three distinct points and a fourth point $z(t)$ depending on $t \in (-\infty, \infty)$ is given by $[z(t), z_1, z_2, z_3] = t$. Locate the positions of $z(t)$ when $t < 0$, $t = 0$, $0 < t < 1$, $t = 1$, $t > 1$, $t \to -\infty$, and $t \to \infty$. This is a way to imagine a circle by a formula of cross-ratio.
2. Use the method of cross ratio to write down the equation of the circles
passing through (a) the points 0, 1, −i (b) the points −1 − i, 0, 1 + i.

3. Suppose that \(\text{Im}[z_1, z_2, z_3, z_4] > 0\). Where is the position of \(z_1\) with respect
to the circle passing through \(z_2, z_3, \) and \(z_4\)? Similarly, consider all the
positions of \(z_j, j = 2, 3, 4\).

4. Find \([f(z_1), f(z_2), f(z_3), f(z_4)]\) in terms of \([z_1, z_2, z_3, z_4]\) for
   (a) \(f(z) = \tau_c(z) = z + c\) where \(c \in \mathbb{C}\);
   (b) \(f(z) = cz\) (a composition of dilation and rotation);
   (c) \(f(z) = c(z) = \overline{z}\);
   (d) \(f(z) = R(z) = 1/z\);
   (e) \(f(z) = \frac{az + b}{cz + d}\) \(a, b, c, d \in \mathbb{C}\).

3.2. Cross Ratio Preseving Functions. In the last exercise above, we see that
there are some complex functions which preserve the cross ratio, that is, they do
not change the cross ratio. Mathematically speaking, for all points \(z_1, z_2, z_3, z_4 \in \mathbb{C}\),

\[ [f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]. \]

Since cross ratios reflect certain properties of the circles, these functions must carry
important geometric information about circles. We would like to see what these
functions are.

**Definition 3.3.** A M"obius transformation on \(\overline{\mathbb{C}}\) is a bijection that preserves the
cross ratio.

The definition is merely an abstract one. At this moment, we do not know which
complex mapping is a M"obius transformation. From the exercise in the previous
section, we have some standard examples of M"obius transformations, the translation,
dilation, rotation, and complex reciprocal. We would like to see if we can find
all the other Möbius transformations from these standard ones. Let us first understand the geometric meaning of these standard mappings. The geometric meaning of the first three types is well understood. To geometrically interpret the complex reciprocal, let us first consider rotation from another point of view.

**Exercise 3.3.** Let $L_1$ and $L_2$ be two straight lines intersecting at a point $p$ at an angle $\alpha$; and $R_1$ and $R_2$ be reflections along the mirrors defined by $L_1$ and $L_2$ respectively. Show that $R_1 \circ R_2$ is a rotation by an angle $2\alpha$ about the point $p$.

From this exercise, two successive reflections in the plane actually give a rotation. We will see that the reciprocal mapping $\mathcal{R}$ is equivalent to two successive “reflections” in the Riemann sphere. Later, we will consider a more general kind of “reflections” using a circular mirror on the Riemann sphere.

Let $z = x + yi \in \mathbb{C}$, then $\text{Re} \left( \frac{1}{z} \right) = \frac{\text{Re} z}{|z|^2}$, $\text{Im} \left( \frac{1}{z} \right) = -\frac{\text{Im} z}{|z|^2}$, and $|\frac{1}{z}| = \frac{1}{|z|^2}$. Therefore,

$$
\mathcal{S}(z) = \left( \frac{2 \text{Re} z}{|z|^2 + 1}, \frac{2 \text{Im} z}{|z|^2 + 1}, |z|^2 - 1 \right),
$$

$$
\mathcal{S} \left( \frac{1}{z} \right) = \left( \frac{\frac{2}{|z|^2} \text{Re} z}{\frac{1}{|z|^2} + 1}, \frac{\frac{2}{|z|^2} \text{Im} z}{\frac{1}{|z|^2} + 1}, \frac{1}{|z|^2} - 1 \right) = \left( \frac{2 \text{Re} z}{1 + |z|^2}, \frac{-2 \text{Im} z}{1 + |z|^2}, 1 - |z|^2 \right).
$$

This shows that if $z \overset{\mathcal{S}}{\mapsto} (p_1, p_2, p_3) \in \mathbb{S}^2$ then $\mathcal{R}(z) \overset{\mathcal{S}}{\mapsto} (p_1, -p_2, -p_3) \in \mathbb{S}^2$. In words, on the sphere, the mapping $\mathcal{R}$ corresponds to a composition of two “reflections”; first reflect the north-south hemispheres and then the left-right. It is also a rotation about an axis going through the points $(1, 0, 0)$ and $(-1, 0, 0)$.

**Proposition 3.4.** Let $f, g : \mathbb{C} \to \mathbb{C}$ be two Möbius transformations. Then so is $g \circ f$. As a consequence, any mapping of the form $z \mapsto \frac{az + b}{cz + d}$ is a Möbius transformation.
**Proof.** First, since both $f$ and $g$ are bijections, so is $g \circ f$. Second, it also preserves cross ratio because

$$[g(f(z_1)), g(f(z_2)), g(f(z_3)), g(f(z_4))] = [f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4].$$

Finally, a mapping of the form $z \mapsto \frac{az + b}{cz + d}$ is clearly a composition of translations, rotations, dilations and $\mathcal{R}$. \hfill \square

**Exercise 3.4.** Fix three complex numbers $\zeta_1$, $\zeta_2$, $\zeta_3$, define a complex function by

$$f(z) \overset{\text{def}}{=} [z, \zeta_1, \zeta_2, \zeta_3].$$

1. What are the values $f(\zeta_1)$, $f(\zeta_2)$, and $f(\zeta_3)$?
2. Also, is it true that $[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$?

Since $(2z + 1)/(z - 1)$ is the same as $(4z + 2)/(2z - 2)$, when we consider mappings of the form $z \mapsto \frac{az + b}{cz + d}$, we usually add the condition $ad - bc \neq 0$. It turns out that there is no other Möbius transformation.

**Theorem 3.5.** Let $f : \mathbb{C} \to \mathbb{C}$ be a Möbius transformation, then there exist complex numbers $a, b, c, d$ with $ad - bc \neq 0$ such that $f(z) = \frac{az + b}{cz + d}$ for all $z \in \mathbb{C}$.

**Proof.** Let $0, 1, \infty$ and an arbitrary point $w \in \overline{\mathbb{C}}$. Since $f$ is a bijection, there are unique points $z_1, z_2, z_3$ and $z \in \overline{\mathbb{C}}$ such that

$$f(z_1) = 0, \quad f(z_2) = 1, \quad f(z_3) = \infty, \quad \text{and} \quad f(z) = w.$$  

Furthermore, $f$ preserves cross ratio, therefore

$$[f(z), 0, 1, \infty] = [z, z_1, z_2, z_3]$$

$$\frac{f(z) - 1}{0 - 1} = \frac{(z - z_2)(z_1 - z_3)}{(z_1 - z_2)(z - z_3)}.$$  

This certainly leads to the form $f(z) = \frac{az + b}{cz + d}$. \hfill \square
Corollary 3.6. Any Möbius transformation is a composition of translation, dilation, rotation, and reciprocal $R$.

Exercise 3.5. (1) In the proof of Theorem 3.5 above, carefully work out the four cases that one of $z_k = \infty$ and none of them equals $\infty$.

(2) Show that if all $a, b, c, d$ are real, then $z \mapsto \frac{az + b}{cz + d}$ maps the real line to the real line and the upper half-space to the upper half-space if $ad - bc > 0$.

(3) Another useful class of Möbius transformations is of the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - az}, \quad a \in \mathbb{C} \text{ with } |a| < 1, \quad \theta \in \mathbb{R}.$$  

Show that $f$ maps the unit circle to itself. Describe its action on the unit circle; inside and outside the circle.

(4) Find the inverse mapping of $f(z) = \frac{az + b}{cz + d}$. Compare this with $2 \times 2$ matrices.

3.3. Types of Möbius Transformations. The technique used in the proof of Theorem 3.5 is actually a general method about Möbius transformations. It will be used again here.

Definition 3.7. A fixed point of a function $f : \mathbb{C} \to \mathbb{C}$ is a point $z_0 \in \mathbb{C}$ such that $f(z_0) = z_0$.

Proposition 3.8. A Möbius transformation is determined by its images on three points. That is, given three points $z_1, z_2, z_3 \in \mathbb{C}$ and their respective images $w_1, w_2, w_3$, there is one and only one Möbius transformation $f$ satisfying

$$w_1 = f(z_1), \quad w_2 = f(z_2), \quad w_3 = f(z_3).$$  

In particular, a Möbius transformation having three fixed points must be the identity map.
**Proof.** As an example, let us look at the case that a Möbius transformation $f$ which has three fixed points. That is, there are

\[ z_1 = f(z_1), \quad z_2 = f(z_2), \quad z_3 = f(z_3). \]

Then, since $f$ preserves cross ratio, for any arbitrary $z \in \mathbb{C}$, it must have

\[ [f(z), z_1, z_2, z_3] = [f(z), f(z_1), f(z_2), f(z_3)] = [z, z_1, z_2, z_3]. \]

This equation gives a formula for $f(z)$ uniquely and it must be $f(z) = z$. In fact, this argument works for the general case, the Möbius transformation sending $z_k$’s to $w_k$’s must have

\[ [f(z), w_1, w_2, w_3] = [z, z_1, z_2, z_3]. \]

This defines one and only one function. \( \square \)

**Exercise 3.6.** Find the Möbius transformation that maps the points $1$, $-1$, $-i$ to the points $1 + i$, $i$, $-1 + i$ respectively.

After we have handled Möbius transformation with three fixed points, we would like to see those with fewer fixed points. It turns out that knowing the number of fixed points of a Möbius transformation is very important to the understanding of its geometric behavior. First, a Möbius transformation is of the form $f(z) = \frac{az + b}{cz + d}$. By solving the equation

\[ z = \frac{az + b}{cz + d}, \]

one gets the fixed points of $f$. As a result, every Möbius transformation must have at least one fixed point. From the above, it can have at most three when it is the identity. So, we only need to consider the cases that it has one or two fixed points.

In our previous examples, a dilation $z \mapsto \rho z$ and a rotation $z \mapsto e^{i\theta} z$ both have two fixed points $0$ and $\infty$; a translation $z \mapsto z + a$ has only $\infty$ as fixed point. In the following, we will see these are the typical examples of Möbius transformations. In
other words, every Möbius transformation behaves similarly as one of them. First, we would like to group similar Möbius transformations together.

**Definition 3.9.** Let \( \mu, f : \mathbb{C} \to \mathbb{C} \) are Möbius transformations. The function \( g = \mu \circ f \circ \mu^{-1} \) is called a conjugate of \( f \).

The concept of conjugation is very common in mathematics. For example, the rotation map \( \rho_{\theta}(z) = e^{i\theta}z \) is rotating about the origin. Then a rotation about another point \( a \in \mathbb{C} \) is actually the conjugated mapping \( \tau_a \circ \rho_{\theta} \circ \tau_a^{-1} \). That is, we first take everything including the point \( a \) back to the origin, do the rotation and then translate everything to \( a \). Usually, a function \( f \) and its conjugate \( \mu f \mu^{-1} \) behave the same in many ways.

Since \( \mu \) is a bijection, its inverse function \( \mu^{-1} \) is defined and \( g \) is well-defined. Moreover, \( g \) is also a Möbius transformation because it does not change cross ratios. It can be easily checked that the fixed points of \( g \) are highly related to those of \( f \).

In fact, \( z_0 \) is a fixed point of \( f \) if and only if \( \mu(z_0) \) is a fixed point of \( g \). Therefore, conjugation does not change the number of fixed points.

**Exercise 3.7.**

1. Find a Möbius transformation which sends \( z_1 \in \mathbb{C} \) to \( \infty \).

2. Let \( z_1, z_2 \in \mathbb{C} \) be different points, find a Möbius transformation which sends \( z_1 \) to \( 0 \) and \( z_2 \) to \( \infty \).

3. Let \( z_1, z_2, z_3 \in \mathbb{C} \) be distinct points, find a Möbius transformation which sends \( z_1 \) to \( 0 \) and \( z_2 \) to \( 1 \) and \( z_3 \) to \( \infty \).

4. Find a Möbius transformation which sends \( 0 \) to \( w_1 \), \( 1 \) to \( w_2 \), \( \infty \) to \( w_3 \).

   Combine this and the preceding one to obtain a Möbius transformation that takes any given points \( z_1, z_2, z_3 \) to \( w_1, w_2, w_3 \).
With the results of these exercises, if \( f \) is a Möbius transformation with one fixed point \( z_0 \), then by choose a suitable \( \mu \), its conjugate \( g \) has the fixed point \( \infty \). Since \( g \) is also a Möbius transformation, it is of the form \( g(z) = \frac{az + b}{cz + d} \). Using \( g(\infty) = \infty \), we have \( c = 0 \). Therefore, \( g(z) = a_1z + b_1 \) after dividing by \( d \). Finally, since \( \infty \) is the only fixed point, we can conclude \( a_1 = 1 \) and \( b_1 \neq 0 \). This leads to the one result of the following classification of Möbius transformations.

**Theorem 3.10.** Any Möbius transformation with exactly one fixed point is conjugate to a translation; anyone having exactly two fixed points is conjugate to either a rotation or \( z \mapsto \lambda z, |\lambda| \neq 0, 1 \) or \( \infty \).

**Exercise 3.8.** Prove the cases of Möbius transformations having two fixed points.

Together with previous results, the action of a Möbius transformation is basically a combination of the dilation, rotation, and translation. The geometric understanding of this fact is further discussed in the appendix.